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A STRAIN-GRADIENT THEORY FOR PRESTRAINED LAMINATES

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Abstract—A simple strain-gradient theory is developed for prestrained laminated materials consisting of thin, stiff reinforcing sheets embedded between thick, soft layers of a matrix material. The theory is used to investigate three types of instability that may occur in massive bodies: global, edge and internal buckling. The predictions for internal buckling are compared with experimental results obtained from a specially constructed model of a laminated body.

INTRODUCTION

IN MANY composite materials the structural nonhomogeneity, or texture, is virtually macroscopic, and may therefore have a significant effect upon the gross behavior of a body. Moreover, this effect cannot be described by the conventional continuum theories—unless, that is, these theories are suitably extended.

A single theory that would encompass the details of all imaginable textures would certainly be quite unworkable. There exist, in fact, numerous "continuum theories of microstructure". These theories are highly specialized and, as a rule, cannot be expected to apply beyond the particular problem in which each theory was originated. For the same reasons, purely formal studies of such theories seem to be rather fruitless.

A direct approach to problems arising from structural nonhomogeneity should start with a definite model of an "elementary cell" that is a small but representative sample of the body. Next, the basic states of the cell are described in terms of appropriate generalized coordinates. The mechanical response can then be characterized by a Lagrangean function $L = K - V$, where K and V are the kinetic energy and strain energy, respectively.

The method outlined so far amounts to using a finite-element model of the composite. The continuum theories of microstructure are based on the same principles as the finite-element method, but with one essential modification: a *smoothing procedure* is used to replace the discrete generalized coordinates by continuous, global deformation fields. As a consequence, the algebraic equations of motion are replaced by differential equations.

The resulting theories are a refinement of the "effective modulus" theories, because they not only predict the gross behavior of a composite but also provide estimates of local deformations and stresses. This is an important feature because realistic failure criteria are likely to require a knowledge of local conditions.

The main limitation of microstructure theories (in comparison to finite-element methods) is that they exclude fields that vary rapidly with respect to the material texture. Therefore, in special regions (e.g. boundary layers) one may have to revert to the finite-element formulation. The shortcomings of a continuum theory are, however, more than compensated for by the simplicity of solutions of instructive problems that would be quite intractable in any other theory which accounts for local nonhomogeneity.

The application of continuum theories of microstructure to dynamics of laminated materials has been explored at some length; a survey of this work is given by Herrmann [1]. A similar approach is also applicable to problems of instability, as shown by Kiusalaas and Jaunzemis [2], Sun [3] and Perkins [4]. The proposed theories, are however, quite complicated, and therefore difficult to apply to nontrivial boundary value problems. In what follows, we consider admissible simplifications of the microstructure theory developed in [2]. In particular, it will be shown that a very simple strain-gradient theory is adequate for predicting the main instabilities that may occur in massive bodies—namely, global, edge and internal buckling. The latter case is verified experimentally using a specially manufactured laminated slab.

We note in passing that similar problems have been considered by Biot [5]. This work uses the notion of couple-stress, and appears to introduce additional, unnecessary simplifications (e.g. incompressibility). Therefore, a quantitative comparison of the results is not possible.

BASIC ASSUMPTIONS

The analysis is restricted to a laminated material consisting of a periodic array of thin, stiff reinforcing sheets and soft matrix layers, as shown in Fig. 1. Both types of layers are taken to be elastic and isotropic. The prestraining is assumed to be a constant compressive strain e in the x_1 direction; the theory will be limited to plane strain conditions in the x_1–x_2 plane.

It is clear that simple theories will emerge only as a consequence of certain simplifying features inherent in the assumed model. This goal of algebraic simplicity places definite restrictions on the admissible nonhomogeneity as well as on the elastic moduli of the constituents. The first restriction that we impose is (cf. Fig. 1)

$$l \gg h. \tag{1}$$

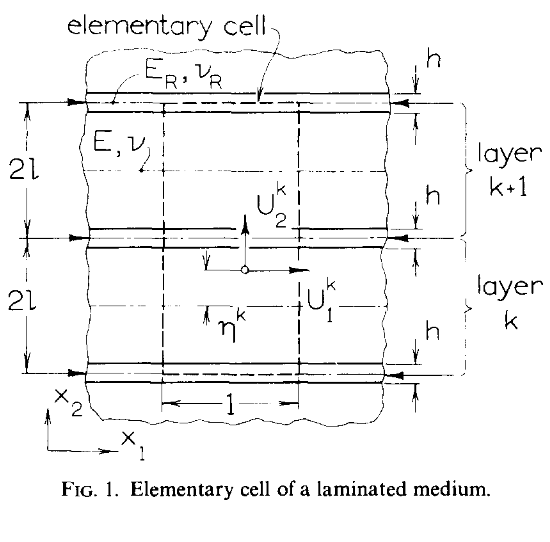

FIG. 1. Elementary cell of a laminated medium.

The above simplification has frequently been used in existing literature, either explicitly or implicitly [2, 5–7]. It agrees with the general design objectives, because it is desirable to keep the volume fraction

$$\alpha = \frac{h}{2l} \tag{2}$$

of the reinforcement small in order to minimize the specific weight of the composite.

In view of (1), the elastic modulus E_R of the reinforcement should be much greater than the elastic modulus E of the matrix material, or else the influence of the reinforcement will be negligible. In fact, we shall adopt the rather stringent condition

$$E_R h^3 \gtrsim E l^3, \tag{3}$$

which now implies

$$E_R \gg E, \qquad E_R h \gg E l. \tag{4}$$

A heuristic interpretation of (3) is that the bending rigidity of a reinforcing sheet should not be small relative to the bending rigidity of a matrix layer. If the reinforcing does not have sufficient resistance to bending, the wavelength of buckling will be small in comparison to the laminate spacing. In this case, a continuum theory is not capable of reproducing the resulting (highly localized) deformations. These findings were first obtained in [6], and have been discussed at a greater length in [2].

According to (4), the prestress is almost entirely carried by the reinforcement. This stress is given by

$$T^0_{11} = -K\text{e}, \qquad K = \alpha \frac{E_R}{1-\nu_R^2}, \tag{5}$$

where K is the "effective modulus", in the x_1 direction, of the composite, and ν_R is the Poisson's ratio of the reinforcement.

The "elementary cell" shown in Fig. 1 consists of two layers. Each layer contains two halves of reinforcements, and a matrix layer of thickness $2l$. The micro-displacement fields U_i^k in a typical matrix layer are assumed to admit the linear approximations

$$U_i^k(x_1, \eta^k) = u_i^k(x_1) + \eta^k \psi_{2i}^k(x_1), \tag{6}$$

where u_i^k are displacements of the middle surface of the matrix. In view of inequality (1), the middle surface displacements V_i^k of the two adjoining reinforcing layers are, approximately

$$V_i^k(x_1) = U_i^k(x_1, \pm l).$$

At this stage a global x_2 coordinate does not occur; rather, there are micro-coordinates η^k and labels k. When calculating micro-strains, differentiation with respect to η^k replaces the differentiation with respect to x_2, and so

$$E_{11}^k = u_{1,1}^k + \eta^k \psi_{21,1}^k, \qquad E_{22}^k = \psi_{22}^k, \qquad E_{12}^k = \tfrac{1}{2}(u_{2,1}^k + \psi_{21}^k + \eta^k \psi_{22,1}^k). \tag{7}$$

Since the displacements must be continuous between the adjacent layers, we must have

$$U_i^k(x_1, l) = U_1^{k+1}(x_1, -l).$$

This condition, when applied to (6), yields

$$\frac{1}{2l}(u_i^{k+1}-u_i^k) = \tfrac{1}{2}(\psi_{2i}^{k+1}+\psi_{2i}^k). \tag{8}$$

We may now introduce various smoothing operations that are compatible with (8). To establish a connection with the gross shape deformation $u_i(x_1, x_2)$, we let

$$\tfrac{1}{2}(u_i^{k+1}+u_i^k) \rightarrow u_i(x_1, x_2). \tag{9}$$

If, in addition, we let

$$\frac{1}{2l}(u_i^{k+1}-u_i^k) \rightarrow u_{i,2}(x_1, x_2), \tag{10}$$

$$\frac{1}{2l}(\psi_{2i}^{k+1}-\psi_{2i}^k) \rightarrow \phi_{2i2}(x_1, x_2), \tag{11}$$

then (8)–(11) provide enough relations for replacing the local fields by global fields. The resulting continuum theory has been explored at length in [2]. Specifically, it was demonstrated that the fundamental buckling modes, shown in Fig. 2, and the corresponding buckling loads agree with the "exact" results of [6] in the long-wavelength approximation. As pointed out previously, however, the theory is too cumbersome to be used for boundary value problems.

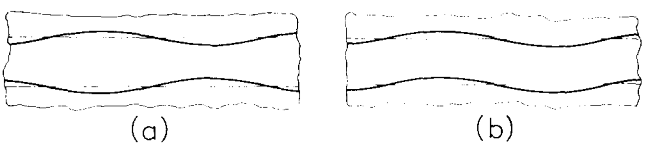

FIG. 2. (a) Extensional buckling mode. (b) Shear buckling mode.

To make the theory more tractable, we introduce the further simplification

$$\phi_{2i2} = u_{i,22}. \tag{12}$$

The price for this simplification is that the theory will not be able to reproduce the "extensional" buckling mode shown in Fig. 2(a). Namely, from (7)–(12) it follows that

$$E_{22}^{k+1}-E_{22}^k = 2lu_{2,22}.$$

Because the global deformation field u_2 varies little from one cell to the next, a theory based on (12) cannot describe deformations in which adjacent reinforcing layers move in opposite directions. Fortunately, this is of no consequence, since the extensional modes, being of short wavelength, will not occur anyway in view of restriction (3) (cf. [2, 6]).

In the present context, there exist two separate strain-gradient effects:

(1) Strain-gradients in the x_1 direction, arising from the bending rigidity of the reinforcing layers (as expressed by the relation 3);

(2) Strain-gradients in the x_2 direction, arising from the mere existence of stratification in that direction.

The latter effect turns out to be quite unimportant. Specifically, it gives rise to shallow boundary-layer effects at boundaries x_2 = const.,† but has virtually no effect on the

† It is also necessary to bear in mind that in such boundary layers a continuum-mechanical approach may not be applicable in the first place.

mechanical response of the body (the observations are briefly discussed in the Appendix).

The preceding remarks suggest a further simplification of the theory: it is permissible to keep only the lowest order derivatives with respect to x_2 in the equations that result from the smoothing operations (8–12). This is equivalent to using simply

$$u_i^k \cong u_i^{k+1} \to u_i(x_1, x_2) \tag{13a}$$

in conjunction with the conventional strain–displacement relations

$$E_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) \tag{13b}$$

for the matrix.

DERIVATION OF THE THEORY

Because the prestress is negligible in the matrix, the incremental strain-energy density is given by the classical formula

$$\phi_M^k = \left(\frac{\lambda}{2}+\mu\right)(E_{11}^k+E_{22}^k)^2+2\mu[(E_{12}^k)^2-E_{11}^kE_{22}^k],$$

where λ, μ are Lame's moduli for the matrix material. The mean strain energy V_M^k stored per unit surface area of the kth matrix layer is then

$$V_M^k = \frac{1}{2l}\int_{-l}^{l} \phi_M^k(\eta^k)\,\mathrm{d}\eta^k. \tag{14}$$

For calculating the incremental strain-energy density per unit area in a reinforcement, we use the conventional formula for a prestressed plate

$$V_R^k = \alpha K\left[(V_{1,1}^k)^2-e(V_{2,1}^k)^2+\frac{h^2}{12}(V_{2,11}^k)^2\right]. \tag{15}$$

For the "elementary cell" consisting of layers k and $k+1$ (cf. Fig. 1), the strain-energy density $V^{k,k+1}$ is given by

$$V^{k,k+1} = \tfrac{1}{2}(V_M^k+V_R^k+V_M^{k+1}+V_R^{k+1}). \tag{16}$$

Substituting (14 and 15) into (16), then making use of the smoothing operations (13), we arrive at the strain-energy function

$$V = \tfrac{1}{2}\Bigg[Ku_{1,1}^2+2\lambda u_{1,1}u_{2,2}+(\lambda+2\mu)u_{2,2}^2+(\mu-eK)u_{2,1}^2 + 2\mu u_{1,2}u_{2,1}+\mu u_{1,2}^2+\frac{Kh^2}{12}u_{2,11}^2\Bigg]. \tag{17}$$

From the function V we can readily obtain the Euler equations

$$Ku_{1,11}+(\lambda+\mu)u_{2,12}+\mu u_{1,22} = 0, \tag{18a}$$

$$(\mu-eK)u_{2,11}+(\lambda+\mu)u_{1,12}+(\lambda+2\mu)u_{2,22}-\frac{Kh^2}{12}u_{2,1111} = 0, \tag{18b}$$

and the boundary conditions

$$T_{11} = Ku_{1,1}+\lambda u_{2,2} = 0 \quad \text{or} \quad u_1 = 0 \text{ at } x_1 = \text{const.} \tag{19a}$$

$$T_{12} = (\mu-eK)u_{2,1}+\mu u_{1,2}-\frac{Kh^2}{12}u_{2,111} = 0 \quad \text{or} \quad u_2 = 0 \text{ at } x_1 = \text{const.} \tag{19b}$$

$$T_{22} = (\lambda+2\mu)u_{2,2}+\lambda u_{1,1} = 0 \quad \text{or} \quad u_2 = 0 \text{ at } x_2 = \text{const.} \tag{19c}$$

$$T_{21} = \mu(u_{2,1}+u_{1,2}) = 0 \quad \text{or} \quad u_1 = 0 \text{ at } x_2 = \text{const.} \tag{19d}$$

$$M = \frac{Kh^2}{12}u_{2,11} = 0 \quad \text{or} \quad u_{2,1} = 0 \text{ at } x_1 = \text{const.} \tag{19e}$$

In deriving (18 and 19) we have used the inequalities $K \gg \lambda$, $K \gg \mu$. Consequently, the matrix cannot be assumed to be incompressible, i.e. λ must remain finite.

SOLUTIONS

The general solution of (18a, b) is

$$u_1 = P\exp(px_1+qx_2), \qquad u_2 = Q\exp(px_1+qx_2). \tag{20}$$

Substitution of (20) in (18a) yields

$$\frac{P}{Q} = -\frac{(\lambda+\mu)pq}{Kp^2+\mu q^2}. \tag{21}$$

Equation (18b), together with (20) and (21) then results in the characteristic equation

$$q^2+(A+B-C)q^2p^2+ABp^4 = 0, \tag{22}$$

where

$$A = \frac{K}{\mu}, \qquad B = \frac{\mu-\mathrm{e}K-(Kh^2/12)p^2}{\lambda+2\mu} \qquad C = \frac{(\lambda+\mu)^2}{\mu(\lambda+2\mu)}. \tag{23}$$

Solution of (22) for q is

$$q^2 = \tfrac{1}{2}p^2\{-(A+B-C)\pm A[1-2(B+C)/A+(B-C)^2/A^2]^{\frac{1}{2}}\}. \tag{24}$$

From the restrictions placed on the properties of the composite it is easily concluded that

$$A \gg |B|, \qquad A \gg C, \tag{25}$$

which enables us to expand the discriminant of (24) in the binomial series

$$q^2 = \frac{p^2}{2}[-(A+B-C)\pm(A-B-C-BC/A+\ldots)].$$

Retaining only the first and second order terms, we get

$$q_1^2 = -(A-C)p^2 \cong -(K/\mu)p^2, \qquad q_2^2 = -Bp^2. \tag{26}$$

It is instructive to solve equations (26) for p (it should be noted that p is also contained in the parameter B):

$$p_1^2 = -q^2/(A-C) \cong -(\mu/K)q^2, \tag{27a}$$

$$p_{2,3}^2 = \left\{\mu - Ke \mp \left[(\mu-Ke)^2 + (\lambda+2\mu)\frac{Kh^2}{3}q^2\right]^{\frac{1}{2}}\right\}\frac{6}{Kh^2} \tag{27b}$$

Equation (27b) illustrates the dependence of buckling modes on the microstructure, i.e. on the parameter h. In particular, we note the degeneracy of the effective modulus theory, obtained by letting $h \to 0$, in which case p_2 becomes indeterminate and $|p_3| \to \infty$ (zero wavelength of buckling).

The three problems that follow are representative of the different types of buckling associated with "massive" bodies, and serve to illustrate the application of the general solution.

Example 1. *Global buckling*

Consider an infinite strip of the laminate, shown in Fig. 3. The boundary conditions correspond to "simple" supports:

$$T_{11} = M = u_2 = 0 \quad \text{at } x_1 = \pm a. \tag{28}$$

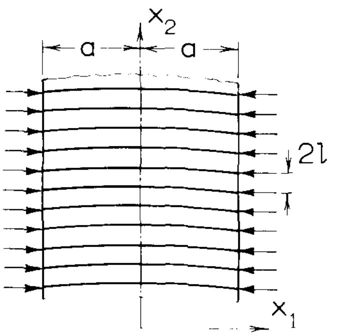

FIG. 3. Global buckling.

The buckling modes are expected to be independent of the x_2-coordinate, i.e. $q = 0$. Then (21) and (27a, b) yield

$$P = p_1 = p_2 = 0, \qquad p_3 = \left\{\frac{12(\mu-Ke)}{Kh^2}\right\}^{\frac{1}{2}}. \tag{29}$$

The boundary conditions (28) can be satisfied only by taking $p_3 = in\pi/(2a)$, where $i = \sqrt{-1}$ and $n = 1, 3, 5 \ldots$, so that

$$u_1 = 0, \qquad u_2 = Q\cos\frac{n\pi x_1}{2a}. \tag{30}$$

Solving the second equation of (29) for e, we obtain

$$e = \frac{\mu}{K} + \frac{1}{12}\left(\frac{n\pi}{2}\frac{h}{a}\right)^2.$$

The minimum value of e is attained with $n = 1$; hence the critical prestrain is

$$e_1 = \frac{\mu}{K} + \frac{\pi^2}{48}\left(\frac{h}{a}\right)^2. \tag{31}$$

We also record for future reference the useful result, obtainable from (23), that $B = 0$ when $e = e_1$.

The first term in (31) equals the critical prestrain obtainable from the effective modulus theory. It corresponds to the vanishing of Biot's "slide modulus" $\mu - eK$ (stiffness with respect to the simple shear $u_1 = 0$, $u_{2,1} = \text{const.}$). The effect of the microstructure is reflected in the second term, which is equal to the Euler buckling strain of a single reinforcing sheet. This result is in agreement with the critical prestrain obtained by Chung and Testa (equation (27a) of [6]).†

Although the effective modulus theory is adequate in calculating the buckling strain for sufficiently small h/a, it is, as pointed out previously, incapable of predicting the buckling modes. With $h = 0$, $\mu - eK = 0$, we find that (18) and (19) are satisfied by $u_1 = 0$, $u_2 = f(x_1)$, where $f(x_1)$ is any function, not necessarily smooth, that satisfies the condition $f(\pm a) = 0$. The bending rigidity of the reinforcement is, therefore, an essential ingredient of the strain-gradient theory, since it endows the mathematical model with unique and smooth mode shapes.

Example 2. Edge buckling

If a free edge is introduced at $x_2 = 0$, as shown in Fig. 4, we expect the buckling deformation to be confined to the vicinity of the edge, and the critical prestrain to be somewhat smaller than in the previous problem, i.e. $B > 0$.

We satisfy the boundary conditions at $x_1 = \pm a$ automatically by taking again $p = i\pi/(2a)$; that is, we use as the solution

$$u_1 = [P_1\, e^{q_1 x_2} + P_2\, e^{q_2 x_2}] \sin\frac{\pi x_1}{2a}, \tag{32a}$$

$$u_2 = [Q_1\, e^{q_1 x_2} + Q_2\, e^{q_2 x_2}] \cos\frac{\pi x_1}{2a}. \tag{32b}$$

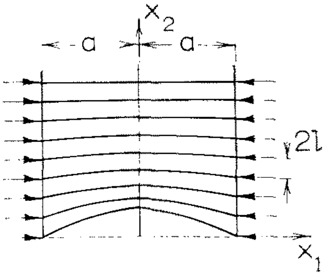

FIG. 4. Edge buckling.

† Since plane stress is used in [6], as opposed to plane strain in the present paper, the results should be compared with zero Poisson's ratios.

Equations (26) then yield

$$q_1 = \pm(A-C)^{\frac{1}{2}}\frac{\pi}{2a}, \qquad q_2 = \pm B^{\frac{1}{2}}\frac{\pi}{2a}. \tag{33}$$

Upon choosing the negative signs, in order to obtain $u_i \to 0$ as $x_2 \to \infty$, we get from (21)

$$\frac{Q_1}{P_1} = -\frac{\pi}{2a}\frac{\mu}{\lambda+\mu}\frac{C}{q_1}, \qquad \frac{Q_2}{P_2} = -\frac{\pi}{2a}\frac{K}{\lambda+\mu}\frac{1}{q_2}. \tag{34}$$

The value of B at buckling and the ratio P_2/P_1 are obtained by substituting the results obtained so far into the boundary conditions at the free edge:

$$T_{22} = T_{21} = 0 \quad \text{at } x_2 = 0, \tag{35}$$

where the expressions for the boundary tractions are obtained from (19c, d). Using simplifications based on (25), the conditions (35) become

$$\left\|\begin{matrix} 1 & \dfrac{\lambda+2\mu}{\lambda+\mu}\dfrac{K}{\mu} \\ 1 & \dfrac{1}{\lambda+\mu}\left(\dfrac{K}{\mu B}\right)^{\frac{1}{2}} \end{matrix}\right\| \left\|\begin{matrix} P_1 \\ P_2 \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \end{matrix}\right\|, \tag{36}$$

from which

$$B = \left(\frac{\mu}{\lambda+2\mu}\right)^2\frac{\mu}{K}, \qquad \frac{P_2}{P_1} = -\frac{\lambda+\mu}{\lambda+2\mu}\frac{\mu}{K}. \tag{37}$$

Equating (23) and the first expression of (37), we obtain the critical prestrain

$$e_2 = e_1 - \frac{\mu}{\lambda+2\mu}\left(\frac{\mu}{K}\right)^2, \tag{38}$$

where e_1 is the critical strain for global buckling (31). The difference between e_1 and e_2 is seen to be negligible in view of the restriction $\mu \ll K$.

The final results for the buckling displacements are

$$u_1 = P_2\left[-\frac{\lambda+2\mu}{\lambda+\mu}\frac{K}{\mu}e^{q_1x_2} + e^{q_2x_2}\right]\sin\frac{\pi x_1}{2a} \tag{39a}$$

$$u_2 = P_2\left(\frac{K}{\mu}\right)^{\frac{1}{2}}\left[-e^{q_1x_2} + \frac{\lambda+2\mu}{\lambda+\mu}\frac{K}{\mu}e^{q_2x_2}\right]\cos\frac{\pi x_1}{2a}, \tag{39b}$$

where

$$q_1 = -\left(\frac{K}{\mu}\right)^{\frac{1}{2}}\frac{\pi}{2a}, \qquad q_2 = -\frac{\mu}{\lambda+2\mu}\left(\frac{\mu}{K}\right)^{\frac{1}{2}}\frac{\pi}{2a}. \tag{39c}$$

In order to exclude rapidly varying displacement fields, we must impose the restriction $|q_i l| \ll 1$. As a consequence, the preceding results are limited to specimens with a sufficiently large dimension a, so that

$$(K/\mu)^{\frac{1}{2}} l/a \ll 1. \tag{40}$$

It is rather interesting to note that in view of $|q_1| \gg |q_2|$ the buckling displacements consist of two boundary effects with greatly different rates of decay.

Example 3. *Internal buckling*

Internal buckling is in a sense the opposite of edge buckling: the boundary conditions prevent surface wrinkling, thereby confining the buckling displacements to the interior of the specimen. An example of internal buckling is shown in Fig. 5. The specimen is infinitely extended in the x_1 direction, and subjected to the "classical" boundary conditions that correspond to rigid, perfectly lubricated surfaces:

$$u_2 = T_{21} = 0 \quad \text{at} \quad x_2 = \pm\frac{b}{2}. \tag{41}$$

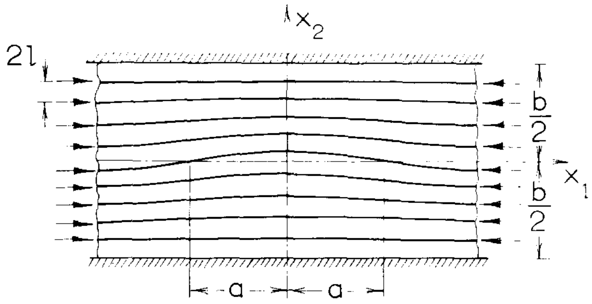

FIG. 5. Internal buckling.

Denoting the wavelength of buckling in the x_1 direction by $4a$, we use, as before $p = i\pi/(2a)$. Consequently equations (32) to (34) of the previous problem remain valid. Due to the constraining effect of the rigid surfaces we expect the critical prestrain to be larger than e_1, which means $B < 0$. We therefore introduce the notation

$$q_2 = \pm i\beta, \qquad \beta^2 = -B\frac{\pi}{2a} > 0. \tag{42}$$

Consider buckling displacements that are antisymmetric about the x_1 axis (intuitively, a larger prestrain would be required to produce the symmetric modes). Then (32a, b) take the form

$$u_1 = (P_1 \sin\beta x_2 + P_2 \sinh q_1 x_2)\sin\frac{\pi x_1}{2a} \tag{43a}$$

$$u_2 = (Q_1 \cos\beta x_2 + Q_2 \cosh q_1 x_2)\cos\frac{\pi x_1}{2a}, \tag{43b}$$

in which the positive values of β and q_1 are used.

It can be shown that the boundary conditions (41) can be satisfied only with

$$P_2 = Q_2 = 0, \qquad \cos\frac{\beta b}{2} = 0. \tag{44}$$

The second equation of (44) yields $\beta b = n\pi$, $n = 1, 3, 5 \ldots$, i.e.

$$B = -\left(2n\frac{a}{b}\right)^2. \tag{45}$$

Substituting the expression for B from (23) in the second equation of (45), we obtain

$$e = \frac{\mu}{K}\left[1+\left(\frac{\pi}{2a}\right)^2\frac{Kh^2}{12\mu}+\frac{\lambda+2\mu}{\mu}\left(2n\frac{a}{b}\right)^2\right]. \tag{46}$$

The minimum value of e occurs with

$$n = 1, \qquad a = \left[\frac{K}{12(\lambda+2\mu)}\right]^{\frac{1}{4}}\left(\frac{\pi bh}{4}\right)^{\frac{1}{2}}, \tag{47}$$

the value of this minimum being

$$e_3 = \frac{\mu}{K}\left\{1+\left[\frac{(\lambda+2\mu)K}{12\mu^2}\right]^{\frac{1}{2}}\frac{2\pi h}{b}\right\}. \tag{48}$$

Finally, we obtain from (34)

$$\frac{Q_1}{P_1} = \frac{\mu}{\lambda+\mu}\left[\frac{3(\lambda+2\mu)}{\mu}\right]^{\frac{1}{4}}\left(\frac{K}{\mu}\right)^{\frac{3}{4}}\left(\frac{2l}{\pi h}\right)^{\frac{1}{2}}. \tag{49}$$

Again we can observe in (47) the degeneracy of the effective modulus theory, namely $a \to 0$ as $h \to 0$.

The theoretical buckling mode was checked experimentally on a specially manufactured specimen consisting of Lexan reinforcing sheets and bubble-rubber matrix. The laminate had the properties

$$h = 0{\cdot}09\text{ in.}, \qquad l = 2{\cdot}78\text{ in.},$$
$$E_R = 457{,}000\text{ psi}, \qquad E = 40\text{ psi},$$
$$\nu_R = 0{\cdot}34, \qquad \nu = 0{\cdot}28,$$

which comply with the restrictions (1, 3), and the overall dimensions

$$c = 30{\cdot}5\text{ in.}, \qquad b = 21{\cdot}6\text{ in.}$$

in the x_1 and x_2 directions, respectively.

Because of the finite length c of the specimen and the simply-supported boundary conditions at $x_1 = \pm c/2$, it is necessary to set $2a = c/m$ in (46), where the integer m is the number of half-waves of buckling in the x_1 direction. As a result, we obtain

$$e = \frac{\mu}{K}\left[1+\frac{K}{12\mu}\left(\frac{\pi h}{c}\right)^2 m^2+\left(\frac{c}{b}\right)^2\frac{\lambda+2\mu}{\mu}\left(\frac{n}{m}\right)^2\right].$$

The critical prestrain is again obtained with $n = 1$; however, the value of m must be computed by trial-and-error. For the specimen described above it was found that the minimum value of e is obtained with $m = 4$, which is in agreement with the experimentally obtained buckling mode shown in Fig. 6.

SUMMARY

We conclude that the simple strain-gradient theory used in the present paper provides a valid idealization of certain composite materials, particularly for predicting the onset

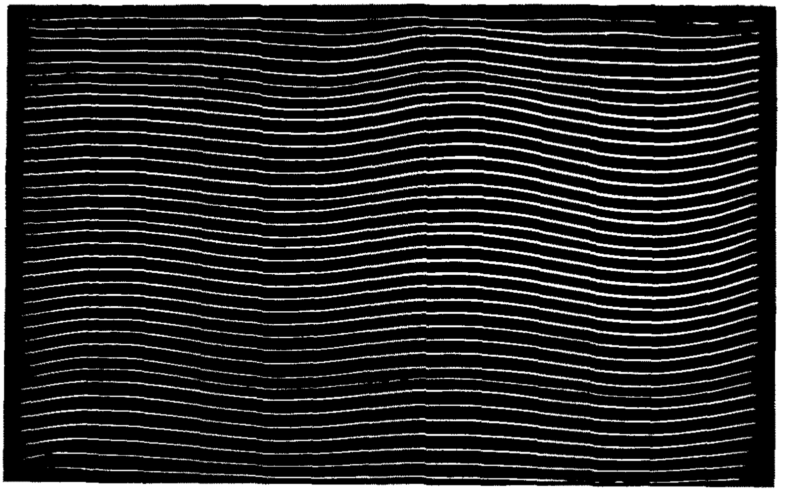

FIG. 6. Experimental model showing internal buckling.

of buckling. Some of the restrictions imposed on the properties of the composite are not intrinsic to the specific theory itself, but reflect the limitations of continuum theories in general. Specifically, if the material properties are such that the buckling deformations are sufficiently smooth, then the simplified strain-gradient approach is just as legitimate as the more elaborate continuum theories.

A further simplification of the equations, namely the effective modulus theory, is not satisfactory, because it leads to indeterminate buckling modes.

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APPENDIX

If we adopt the more elaborate smoothing operations (8–12), the strain-energy function becomes

$$
\begin{aligned}
V = \frac{1}{2}\Bigg[& Ku_{1,1}^2 + 2\lambda u_{1,1}u_{2,2} + (\lambda+2\mu)u_{2,2}^2 \\
& + (\mu - eK)u_{2,1}^2 + 2\mu u_{1,2}u_{2,1} + \mu u_{1,2}^2 + \frac{Kh^2}{12}u_{2,11}^2\Bigg] \\
& + \tfrac{1}{2}l^2\Bigg[2Ku_{1,12}^2 + 2\lambda u_{1,12}u_{2,22} + (\lambda+2\mu)u_{2,22}^2 \\
& + (\tfrac{4}{3}\mu - 2eK)u_{2,12}^2 + 2\mu u_{2,12}u_{1,22} + \mu u_{1,22}^2 + \frac{Kh^2}{6}u_{2,112}^2\Bigg] \\
& + \tfrac{1}{2}l^4\Bigg[Ku_{2,122}^2 + (\tfrac{1}{3}\mu - eK)u_{2,122}^2 + \frac{Kh^2}{12}u_{2,1122}^2\Bigg].
\end{aligned}
\tag{A-1}
$$

The resulting Euler equations are

$$
(1-l^2\partial_2^2)[K(1-l^2\partial_2^2)u_{1,11} + (\lambda+\mu)u_{2,12} + \mu u_{1,22}] = 0 \tag{A-2a}
$$

$$
\begin{aligned}
(1-l^2\partial_2^2)\Bigg\{&[\mu(1-\tfrac{1}{3}l^2\partial_2^2) - eK(1-l^2\partial_2^2)]u_{2,11} + (\lambda+\mu)u_{1,12} \\
&+ (\lambda+2\mu)u_{2,22} - \frac{Kh^2}{12}(1-l^2\partial_2^2)u_{2,111}\Bigg\} = 0,
\end{aligned}
\tag{A-2b}
$$

where $\partial_2 = \partial/\partial x_2$.

A general solution of (A-2a, b) has the form

$$
u_1 = \left(\frac{P}{1-l^2q^2}e^{qx_2} + R_1\, e^{-x_2/l} + R_2\, e^{x_2/l}\right)e^{px_1} \tag{A-3a}
$$

$$
u_2 = \left(\frac{Q}{1-l^2q^2}e^{qx_2} + R_3\, e^{-x_2/l} + R_4\, e^{x_2/l}\right)e^{px_1}. \tag{A-3b}
$$

Substitution of (A-3) in (A-2) will yield the characteristic equation that has the same form as (22), but now

$$
A = \frac{K}{\mu - Kp^2l^2}, \tag{A-4a}
$$

$$
B = \frac{\mu - eK - (Kh^2/12)p^2}{\lambda + 2\mu - [\frac{1}{3}\mu - eK - (Kh^2/12)p^2]p^2l^2}, \tag{A-4b}
$$

$$
C = \frac{(\lambda+\mu)^2}{(\mu - Kp^2l^2)\{\lambda + 2\mu - [\frac{1}{3}u - eK - (Kh^2/12)p^2]p^2l^2\}}. \tag{A-4c}
$$

It is not difficult to see that equations (A-4) can be reduced to the previous result (23) by enforcing the smoothness requirement $|ql| \ll 1$, which, according to (26), also implies $(K/\mu)^{\frac{1}{2}}|pl| \ll 1$. Consequently, equations (A-3) differ from the general solution of the simplified theory (20) only by the addition of the terms $R_i \exp(\pm x_2/l)$. Because these additional terms give rise to a boundary-layer effect at boundaries x_2 = const., their effect would be confined to within a few laminate thicknesses from the boundaries, and not influence the overall response of the laminate. This result can be also established by applying the method of singular perturbations to the equations (A-2) (cf. [8]).

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Абстракт—В работе дается итеративный метод конечного элемента для расчета конструкции на минимум веса, с точки зрения ограничения выпучивания. Выводится переработанное уравнение, путем использования критерия на минимум веса, в противоположности к методу численного поиска. Затем, можно заниматься проблемами, которые отличаются существованием двух основных форм выпучивания для расчета на минимум веса. Иллюстрируется применение метода задачами расчета балки и прямоугольной рамы.